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EXPANSION PROBLEMS FOR HERMITION MATRICES WHILE MAINTAINING THE EIGENVALUES

SUNG-SOO PYO

ABSTRACT. For given Hermition matrix A, in this paper, we will study the existence of an expansion Hermition matrix $\hat{A}$ of A while maintaining the eigenvalues of A. For any proper subset $\{\lambda_{j1}, \lambda_{j2}, \cdots, \lambda_{jk}\}$ of the spectrum $\sigma(A) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of A, we explicitly show the existence of an expansion $\hat{A}$ of A whose spectrum contains all $\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{jk}$. Moreover, we show that there exists a 2-step expansion $\hat{A}$ of A whose spectrum contains all eigenvalues of A, that is, $\sigma(A) \subset \sigma(\hat{A})$.

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1. INTRODUCTION

For an $n \times n$ Hermition matrix A, we call on $(n+1) \times (n+1)$ matrix B a *1-step expansion matrix* of A if there exists an n-dimensional vector $\mathbf{v}$ and a real number a such that B is of the form

$$B = \begin{bmatrix} A & \mathbf{v} \\ \mathbf{v}^T & a \end{bmatrix}.$$

In the same sense, if A is an $n \times n$ principal submatrix of $(n+k) \times (n+k)$ matrix F, then F can be considered an expansion matrix of A. That is, there exist $n \times k$ matrix C and $k \times k$ Hermition D such that F has the form

$$F = \begin{bmatrix} A & C \\ C^T & D \end{bmatrix}.$$

In this case, we say that F is a k-th expanded matrix from A.

As usual, we denote the spectrum of A by $\sigma(A) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Study on the relevance of the eigenvalues of A and the eigenvalues of an expanded matrix $\hat{A}$ of the matrix A are one of main research subjects in the matrix theory.

One of most well known fact is the interlacing theorem, inequalities between eigenvalues of a Hermition matrix and eigenvalues of its expanded matrix [?].

Let A be an $n \times n$ Hermition matrix with eigenvalues $\sigma(A) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ and let $\hat{A}$ be an expanded matrix of A with eigenvalues $\sigma(\hat{A}) = (\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_{n+1})$. Since A and $\hat{A}$ are Hermition, their eigenvalues are all real. So we can arrange the eigenvalues of A in the order $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and the eigenvalues of $\hat{A}$ in the order $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_{n+1}$, $i = 1, 2, \cdots, n$. The interlacing inequalities for a Hermition matrix say that

$$\hat{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_2 \leq \lambda_2 \cdots \leq \lambda_n \leq \hat{\lambda}_{n+1}. \tag{1}$$

In 1981, C. R. Johnson and H. A. Robinson studied eigenvalue inequalities for a Hermition matrix and its principal submatrices using the inequalities between roots of a polynomial and the roots of the derivative [5].

In [7], F. C. Silva gave an existence condition of matrix A For given c_1, c_2, $\ldots, c_{m+n}$, there exist matrices A_{12} and A_{21} such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

has eigenvalues c_1, c_2, $\ldots, c_{m+n}$.

This result extended by G. Cravo in 2009 [1].

In 1993, authors showed that the problem of determining when a normal $n \times n$ matrix A exists satisfying $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(A_i) = \{\lambda_{i1}, \ldots, \lambda_{in-1}\}$, $i = 1, \ldots, n$, is equivalent to determining whether a certain $n \times n$ matrix constructed from the λ_i, λ_{ij} is unistochastic. But they used very strong conditions as if A has eigenvalues c when A's characteristic polynomial conditions [6].

In [3], author gave an elementary proof of (1) with an equality condition.

In this paper, we are going to study spectral properties of Hermition matrices as an inverse eigenvalue problem. We set up the following questions first.

Q1 *For a given subset $\{\lambda_{i1}, \ldots, \lambda_{ik}\}$ of $\sigma(A)$, expanding matrix A so that the spectrum of expanded matrix $\hat{A}$ contains given subset of eigenvalues $\{\lambda_{i1}, \ldots, \lambda_{ik}\}$ of A, that is, $\{\lambda_{i1}, \ldots, \lambda_{ik}\} \subset \sigma(\hat{A})$.*

2. Expanding Hermition matrix while maintaining the eigenvalues

A square matrix A is said to be *reducible* if there exists a permutation matrix P such that

$$P^T AP = \begin{bmatrix} A_{11} & O \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are non-vacant square matrices. Otherwise, we say *irreducible* if it is not reducible.

If a Hermition matrix A is reducible, then A can be expressed a direct sum of two principal submatrices $A = A_{11} \oplus A_{22}$.

Because of very simple reason, $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$ if A is reducible, from now on, we only consider irreducible cases.

At first, we are going to show that there is no $(n+1) \times (n+1)$ matrix $\hat{A}$ which is an expanded matrix of A of order n and whose spectrum contains all eigenvalues of A, that is, $\sigma(A) \not\subseteq \sigma(\hat{A})$.

Theorem 2.1. *Let $A = [a_{ij}]$ be an irreducible Hermition matrix of order n and let $\hat{A}$ be any 1 -step expanded Hermition matrix of A by bordering a* *row and a column. Then $\sigma(A) \not\subseteq \sigma(\hat{A})$. That is, $\sigma(\hat{A})$ can not contain all eigenvalues of A.*

Proof. Let an expanded matrix $\hat{A}$ of A be of the form

$$\hat{A} = \begin{bmatrix} A & \mathbf{y}^* \\ \mathbf{y} & a \end{bmatrix}.$$

Then the characteristic equation $p_{\hat{A}}(\lambda)$ of $\hat{A}$ is

$$p_{\hat{A}}(\lambda) = (\lambda - a)p_A(\lambda) + f(\lambda)$$

where $p_A(\lambda)$ is the characteristic polynomial of A and $f(\lambda)$ is a polynomial of degree $n-1$. If $\sigma(A) \subset \sigma(\hat{A})$, then each eigenvalue of A is a root of $p_{\hat{A}}(\lambda) = 0$. Therefore, each eigenvalues of A is a root of $f(\lambda) = 0$ with the same multiplicity of the multiplicity of eigenvalues of A. This is impossible. ■

We are going to prove this theorem one more time after Theorem 2.2 (Theorem 2. in [3]). Theorem 2.2 says more simple reason of validity than Theorem 2.1.

From Theorem 2.1, considering all the multiplicity of each eigenvalue, we know that the spectrum of simple extended matrix can not contain all eigenvalues of the original matrix. If we expand sufficiently many steps, then we can obtain an expanded matrix of A whose spectrum contains all of the eigenvalues of A. The following example shows this.

Example 1. For a given $n \times n$ matrix A, let's consider following

$$\hat{A} = \begin{bmatrix} A & \mathbf{v}_1 & O \\ \mathbf{w}_1^T & 0 & \mathbf{w}_2^T \\ O & \mathbf{v}_2 & A \end{bmatrix}$$

where $\mathbf{v}_i, \mathbf{w}_j, (i, j = 1, 2)$ are arbitrary n-dimensional vectors. Then Laplace expansion of determinant of $\hat{A}$ along the $(n+1)$th row, each term contains $\det A$. This property holds in the expansion of $\det(tI - \hat{A})$. Thus, $\sigma(A)$ is a subset of $\sigma(\hat{A})$. In fact, if $A\mathbf{x} = \lambda_1\mathbf{x}$, then there exist real numbers c_1 and c_2 such that $\hat{A}\hat{\mathbf{x}} = \lambda_1\hat{\mathbf{x}}$ where $\hat{\mathbf{x}} = (c_1\mathbf{x}, 0, c_2\mathbf{x})^T$. ■

In the Example 1., any given Hermition can be expanded to a matrix which contains all eigenvalues of original one's. Therefore the following question arise.

Q2 *Given Hermition A, what is the least step expansion so that the spectrum of expanded matrix contains all the eigenvalues of A ?*

Before to give an answer for this question Q2, let's observe the following.

For an $n \times n$ Hermition A, let $\mathbf{x}$ be an eigenvector of A with respect to eigenvalue λ and let n-dimensional vector $\mathbf{v}$ be orthogonal to $\mathbf{x}$. That is,

$$A\mathbf{x} = \lambda\mathbf{x} \text{ and } \mathbf{v}^T\mathbf{x} = 0.$$

Then the following holds.

$$\begin{bmatrix} A & \mathbf{v} \\ \mathbf{v}^T & a \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$$

This can be founded in [3], author presented the following theorem.

Theorem 2.2. *Let*

$$A = \begin{bmatrix} a & \mathbf{y}^* \\ \mathbf{y} & B \end{bmatrix}$$

be a Hermition matrix, and let β be an eigenvalue of B of multiplicity p. Then β is an eigenvalue of A of multiplicity at least p if and only if $\mathbf{y}$ is orthogonal to the eigenspace of B corresponding to the eigenvalue β.

One may guess that the orthogonality between A_{12} and eigenspaces is not only sufficient condition but also necessary condition. But the Example 1 shows that this is not true. And now we can prove the Theorem 2.1.

Proof. (of Theorem 2.1) Let A be a Hermition matrix and let

$$\hat{A} = \begin{bmatrix} A & \mathbf{w} \\ \mathbf{w}^T & a \end{bmatrix}.$$

Suppose the spectrum of $\hat{A}$ contains all eigenvalues of A, then $\mathbf{w}$ is orthogonal to the all eigenspaces of A. This lead in to that the n-dimensional vector $\mathbf{w}$ is orthogonal to the n linearly independent vectors. So this is impossible. ∎

Theorem 1 can be extended to the following.

Corollary 2.3. *For any $n \times n$ matrix A, for any proper subset $\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}\} \subsetneq \sigma(A)$ and $k < n$, there exists one step expansion $\hat{A}$ of A whose spectrum contains k eigenvalues of A, $\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}\} \subset \sigma(\hat{A})$.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be eigenvectors of A corresponding to eigenvalues $\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}\}$ respectively. Let $\mathbf{v}$ be an n dimensional real vector orthogonal to all eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then, for any real a,

$$\begin{bmatrix} A & \mathbf{v} \\ \mathbf{v}^T & a \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{o} \end{bmatrix} = \begin{bmatrix} \lambda_{ik}\mathbf{x}_k \\ \mathbf{o} \end{bmatrix} = \lambda_{ik} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{o} \end{bmatrix}$$

Therefore $\mathbf{x}_1 \oplus \mathbf{o}, \ldots, \mathbf{x}_k \oplus \mathbf{o}$ are eigenvectors of $\hat{A}$ corresponding to eigenvalues $\{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}\}$ respectively. ∎

This corollary 3 is the answer to the quaestion Q1.

Now, we would like to say the existence of extension whose spectrum contains all eigenvalues of it's principal minor. Before talking about the extension, let's observe the characteristic polynomial of an extension of A.

Let $\hat{A}$ be a k-step extension of a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then the characteristic polynomial of $\hat{A}$ is the following

$$\det(tI - \hat{A}) = \begin{vmatrix} tI - D & -A_{12} \\ -A_{12}^T & tI - B \end{vmatrix} = \det(tI - D)\det(tI - B) + f(t).$$

Therefore, the spectrum of $\hat{A}$ contains all eigenvalues of D if and only if $f(t)$ is the zero polynomial or $f(\lambda_i) = 0$ for $i = 1, \ldots, n$.

Theorem 2.4. *Let $A \in M_n$ be a given irreducible Hermition matrix. Then there exists an 2 step expansion matrix $\hat{A}$ whose spectrum contains all eigenvalues of A.*

Proof. We are going to prove this explicitly. Since A is Hermition, there exists a unitary matrix U such that $U^T AU = D$ where D is the diagonal matrix whose diagonal entries are eigenvalues of A, $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Let's set two step expansion matrix $\tilde{A}$ as follows :

$$\tilde{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \lambda_n & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \lambda_1 \end{bmatrix}.$$

Then we can easily calculate the next characteristic polynomial of $\tilde{A}$.

$$p_{\tilde{A}}(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)(t^2 - 2\lambda_1 t + \lambda_1^2 - 2).$$

Therefore the spectrum of $\tilde{A}$ contains all the eigenvalues of A, $\lambda_i, i = 1, \ldots, n$. From this $\tilde{A}$, we get $\hat{A}$

$$\hat{A} = \begin{bmatrix} U & O \\ O & I \end{bmatrix} \tilde{A} \begin{bmatrix} U^T & O \\ O & I \end{bmatrix}$$

Therefore

$$\hat{A} = \begin{bmatrix} A & \mathbf{x} & \mathbf{o} \\ \mathbf{x}^T & \lambda_1 & 1 \\ \mathbf{o}^T & 1 & \lambda_1 \end{bmatrix} \tag{2}$$

where $\mathbf{x}$ is an eigenvector of A with respect to the eigenvalue λ_1 and $\mathbf{o}$ is the n-dimensional zero vector. Since A is irreducible and $\mathbf{x} \neq \mathbf{o}$, $\hat{A}$ is irreducible. ■

The equation (2) specifically shows what type of expanded matrix is.

Corollary 2.5. *Let A be an Hermition matrix of order n and $\hat{A}$ be of the form*

$$\hat{A} = \begin{bmatrix} A & c\mathbf{x} & \mathbf{o} \\ c\mathbf{x}^T & \lambda_1 & 1 \\ \mathbf{o}^T & 1 & \lambda_1 \end{bmatrix}$$

where λ_1 is an eigenvalue of A with corresponding normalized eigenvector $\mathbf{x}$ and c is any real constant. Then $\sigma(A) \subset \sigma(\hat{A})$ and $\lambda_1 \pm \sqrt{c^2+1} \in \sigma(\hat{A})$.

Proof. Assume that λ is an eigenvalue of A different λ_1, and it's corresponding eigenvector is $\mathbf{y}$. Then $\mathbf{y}^T\mathbf{x} = \mathbf{x}^T\mathbf{y} = 0$.

$$\begin{bmatrix} A & c\mathbf{x} & \mathbf{o} \\ c\mathbf{x}^T & \lambda_1 & 1 \\ \mathbf{o}^T & 1 & \lambda_1 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{y} \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

And since $\mathbf{x}$ is an eigenvector of A corresponding to λ_1,

$$\begin{bmatrix} A & c\mathbf{x} & \mathbf{o} \\ c\mathbf{x}^T & \lambda_1 & 1 \\ \mathbf{o}^T & 1 & \lambda_1 \end{bmatrix} \begin{bmatrix} -\mathbf{x} \\ \mathbf{o} \\ c \end{bmatrix} = \lambda_1 \begin{bmatrix} -\mathbf{x} \\ \mathbf{o} \\ c \end{bmatrix}.$$

Therefore $\sigma(A) \subset \sigma(\hat{A})$.

Let's observe the following

$$\begin{bmatrix} A & c\mathbf{x} & \mathbf{o} \\ c\mathbf{x}^T & \lambda_1 & 1 \\ \mathbf{o}^T & 1 & \lambda_1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ a \\ b \end{bmatrix} = \begin{bmatrix} (\lambda_1 + ac)\mathbf{x} \\ c + a\lambda_1 + b \\ a + b\lambda_1 \end{bmatrix}.$$

Then possible eigenvalue is $\lambda_1 + ac$, so we can derive next equation

$$\begin{aligned} c + a\lambda_1 + b &= (\lambda_1 + ac)a \\ a + b\lambda_1 &= (\lambda_1 + ac)b \end{aligned}$$

From this, we get $a = \pm\sqrt{1 + (\frac{1}{c})},\ b = \frac{1}{c}$. So, we reached the conclusion.
■

Theorem 4 is an answer for the question Q2. It is easy to check that this results are also applicable to the diagonalizable matrix of real spectrum.

Corollary 2.6. *Let $B \in M_n$ be any irreducible diagonalizable matrix of real spectrum. Then there exists an 2 step expansion matrix $\hat{B}$ whose spectrum contains all eigenvalues of A.*

3. **Interlacing Eigenvalues and Additional Remark**

Now we want to observe interlacing of eigenvalues between expanded matrix and original matrix.

$$\hat{\lambda}_1 = \lambda_1 \le \hat{\lambda}_2 \le \lambda_2 \le \cdots \le \lambda_n \le \hat{\lambda}_{n+1},$$

By one step extension, we can take matrix which has common eigenvalue λ_i with its principal minor's. So one can ask whether $\hat{\lambda}_{i-1} = \lambda_i$ or $\lambda_i = \hat{\lambda}_{i+1}$. But the following example shows that this is not valid for all cases.

Example 2. For a symmetric matrix A,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

Then the eigenvalues of A is

$$\sigma(A) = (1 - \sqrt{3}, 1, 1 + \sqrt{3}),$$

and corresponding eigenvectors are

$$\begin{aligned} \mathbf{x}_1 &= (-2 - \sqrt{3}, 1 + \sqrt{3}, 1)^T \\ \mathbf{x}_2 &= (1, 1, 1)^T \\ \mathbf{x}_3 &= (-2 + \sqrt{3}, 1 - \sqrt{3}, 1)^T. \end{aligned} \tag{3}$$

For observation of interlacing eigenvalues between A and it's expansion matrix, we set $\lambda_1 \approx -0.73, \lambda_2 = 1, \lambda_3 \approx 2.73$.

At first, let's constructing expanded matrix B of A which has eigenvalue 1. The eigenvector of A corresponding 1 is $\mathbf{x}_2 = (1,1,1)^T$. So there are many possibility to choice vectors for containing 1 as eigenvalues of expanded matrix. Next two matrices are expanded matrices whose spectrum contains 1.

$$B = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

Then spectrum of B and C is

$$\sigma(B) \approx (-1.51, 0.42, 1, 3.08)$$

$$\sigma(C) \approx (-2.79, 1, 1.79, 3).$$

So interlacing eigenvalues are as follows

$$\begin{array}{llllll} -1.51 & < \lambda_1 & < 0.42 & < 1 = \lambda_2 & < \lambda_3 & < 3.08 \\ -2.79 & < \lambda_1 & < 1 = \lambda_2 & < 1.79 & < \lambda_3 & < 3 \end{array}$$

There is no further comment here. Just because λ is the order in which the original value 1 is ranked aleady.

Let's construct an expansion matrix $\tilde{A}$ of A whose spectrum contains $1, 1-\sqrt{3}$. At first, we have to find a vector which is orthogonal to $\mathbf{x}_2$ and $\mathbf{x}_3$. The next $\mathbf{v}$ is taken by outer product of $\mathbf{x}_2$ and $\mathbf{x}_3$.

$$\mathbf{v} = (-\sqrt{3}, -3-\sqrt{3}, 3+2\sqrt{3})^T$$

$$\tilde{A}_1 = \begin{bmatrix} 0 & 1 & 0 & -\sqrt{3} \\ 1 & 1 & -1 & -3-\sqrt{3} \\ 0 & -1 & 2 & 3+2\sqrt{3} \\ -\sqrt{3} & -3-\sqrt{3} & 3+2\sqrt{3} & 0 \end{bmatrix},$$

Then spectrum of $\tilde{A}$ is

$$\lambda_1 = \frac{1}{2}\left(1+\sqrt{3}+\sqrt{74(2+\sqrt{3})}\right) \approx 9.68$$

$$\lambda_2 = \frac{1}{2}\left(1+\sqrt{3}-\sqrt{74(2+\sqrt{3})}\right) \approx -6.94$$

$$\lambda_3 = 1$$

$$\lambda_4 = 1-\sqrt{3} \approx -0.73$$

Their inequality is as follows,

$$-6.94 < \lambda_1 \approx -0.73 < \lambda_2 < \lambda_3 < 9.68.$$

Let's take a look at the different examples to know the order of eigenvalues of expanding matrix.

$$\tilde{A}_2 = \begin{bmatrix} 0 & 1 & 0 & \sqrt{3} \\ 1 & 1 & -1 & -3+\sqrt{3} \\ 0 & -1 & 2 & 3-2\sqrt{3} \\ \sqrt{3} & -3+\sqrt{3} & 3-2\sqrt{3} & 0 \end{bmatrix} \quad \tilde{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Then,

$$\sigma(\tilde{A}_2) = (1+\sqrt{3}, \frac{1}{2}\left(1-\sqrt{3}-\sqrt{74(2-\sqrt{3})}\right), \frac{1}{2}\left(1-\sqrt{3}+\sqrt{74(2-\sqrt{3})}\right), 1).$$

$$\sigma(\tilde{A}_3) = (1+\sqrt{3}, \frac{1}{2}(1+\sqrt{13}), \frac{1}{2}(1-\sqrt{13}), 1-\sqrt{3})$$

So in the case $\tilde{A}_2$ interlacing is

$$\sigma(\tilde{A}_2) \approx (2.73, -2.59, 1.86, 1)$$
$$-2.59 < \lambda_1 < 1 = \lambda_2 < 1.86 < 2.73 \approx \lambda_3,$$

and in the case $\tilde{A}_3$ interlacing is

$$\sigma(\tilde{A}_3) \approx (2.73, 2.30, -1.30, -0.73)$$
$$-1.30 < -0.73 = \lambda_1 < 2.30 < 2.73 \approx \lambda_3.$$

Here, we expand matrix for which contains two original eigenvalues. And we were to find some interlacing relations. But we could not found any relation in generally.

Now let's construct two step expansion of A which contains all eigenvalues of A. Therefore we construct nonsingular S for diagonalization A as follows

$$S = \begin{bmatrix} 1 & -2+\sqrt{3} & -2-\sqrt{3} \\ 1 & 1-\sqrt{3} & 1+\sqrt{3} \\ 1 & 1 & 1 \end{bmatrix}.$$

All column vectors of S are orthogonal to each other, so we can drive diagonalization unitary matrix U for A as follows

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2+\sqrt{3}}{3-\sqrt{3}} & \frac{-2-\sqrt{3}}{3+\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-\sqrt{3}}{3-\sqrt{3}} & \frac{1+\sqrt{3}}{3+\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3-\sqrt{3}} & \frac{1}{3+\sqrt{3}} \end{bmatrix}.$$

By Theorem 4, we can derive

$$\hat{A} = \begin{bmatrix} S^{-1} & O \\ O & I_2 \end{bmatrix} \begin{bmatrix} A & \mathbf{v} \\ \mathbf{v}^T & D \end{bmatrix} \begin{bmatrix} S & O \\ O & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2+\sqrt{3}}{3-\sqrt{3}} & \frac{-2-\sqrt{3}}{3+\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1-\sqrt{3}}{3-\sqrt{3}} & \frac{1+\sqrt{3}}{3+\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{3-\sqrt{3}} & \frac{1}{3+\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1+\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 1-\sqrt{3} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{-2+\sqrt{3}}{3-\sqrt{3}} & \frac{1-\sqrt{3}}{3-\sqrt{3}} & \frac{1}{3-\sqrt{3}} & 0 & 0 \\ \frac{-2-\sqrt{3}}{3+\sqrt{3}} & \frac{1+\sqrt{3}}{3+\sqrt{3}} & \frac{1}{3+\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 1 & 1 & -1 & \frac{1}{\sqrt{3}} & 0 \\ 0 & -1 & 2 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The spectrum of $\sigma(\hat{A}) = (1+\sqrt{3}, 1+\sqrt{2}, 1, 1-\sqrt{2}, 1-\sqrt{3})$
Therefore interlacing of eigenvalues is as follow

$$1-\sqrt{3} = \lambda_1 < 1-\sqrt{2} < 1 = \lambda_2 < 1+\sqrt{2} < 1+\sqrt{3} = \lambda_3$$

If we take as

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1+\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 1-\sqrt{3} & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

then $\hat{A}$ is of the form

$$\hat{A}_2 = \begin{bmatrix} 0 & 1 & 0 & \frac{2}{\sqrt{3}} & 0 \\ 1 & 1 & -1 & \frac{2}{\sqrt{3}} & 0 \\ 0 & -1 & 2 & \frac{2}{\sqrt{3}} & 0 \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 1 & 1 \\ 0 & 0 & 0 & 1 & 1. \end{bmatrix}$$

and $\sigma(\hat{A}_2) = (1+\sqrt{5}, 1+\sqrt{3}, 1, 1-\sqrt{3}, 1-\sqrt{5})$.
In this case, interlacing of eigenvalues is

$$1-\sqrt{5} < 1-\sqrt{3} = \lambda_1 < 1 = \lambda_2 < 1+\sqrt{3} = \lambda_3 < 1+\sqrt{5}$$

Thus, the order of eigenvalues is determined by chosen elements of matrix and not fixed under expanded matrix form.

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References

[1] G. Cravo, *Matrices with Prescribed Eigenvalues and Prescribed Submatrices*, Linear and Multilinear Algebra. 57(2009) no. 7, 737–748.
[2] R. A. Horn and C. R. Johnson, *Matrix analysis* Cambridge University Press, New York, 1985.
[3] S-G. Hwang, *Cauchy's interlace theorem for eigenvalues of Hermitian matrices*, Amer. Math. Monthly 111 (2004) no. 2, 157–159.
[4] C. R. Johnson, B. Kroschel, M. Omladic, *Eigenvalue multiplicities in Principal Submatrices*, Linear Algebra Appl. 390(2004) 111–120.
[5] C. R. Johnson, H. A. Robinson, *Eigenvalue Inequalities for Principal Submatrices*, Linear Algebra Appl. 37(1981) 11–22.
[6] P. Nylen, T.-Y. Tam, F. Uhlig, *On the Eigenvalues of Principal Submatrices of Normal, Hermition and Symmetric matrices*, Linear and Multilinear Algebra, 36(1993) 69–78.
[7] F. C. Silva, *Matrices with Prescribed Eigenvalues and Principal Submatrices*, Linear Algebra Appl. 92(1987) 241–250.

Department of Mathematics Education, Silla University, Busan, Korea
E-mail address: ssoopyo@silla.ac.kr